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Multivariate Trigonometric B-Splines

PER ERIK KOCH

Institute of Informatics, University of Oslo, Oslo, Norway

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A relation between univariate trigonometric and polynomial B-splines is extended to higher dimensions and used to construct a class of multivariate trigonometric B-splines from the multivariate polynomial ones. These new functions are trigonometric splines in each variable, but they are not tensor product splines. They are smooth and have local non-polygonal support. They possess a recurrence relation similar to that of the multivariate polynomial B-splines. © 1988 Academic Press, Inc.

1. INTRODUCTION

In the next section we find a relation between univariate trigonometric and polynomial B-splines. In Section 3 we extend this relation to higher dimensions and use it to construct a class of multivariate trigonometric B-splines from the multivariate polynomial ones. Some further generalizations are given in Section 4.

Let us first recall some properties of univariate and multivariate polynomial B-splines.

Denote the (unnormalized) univariate polynomial B-spline corresponding to the knots $x^0 \leqslant x^1 \leqslant \cdots \leqslant x^n$ by $Q(\cdot \mid x^0, ..., x^n)$. By de Boor [1],

$$Q(x \mid x^0, ..., x^n) = \frac{(x - x^0)\, Q(x \mid x^0, ..., x^{n-1}) + (x^n - x)\, Q(x \mid x^1, ..., x^n)}{x^n - x^0}. \quad (1)$$

For $n = 1$,

$$Q(x \mid x^0, x^1) = \begin{cases} 1/(x^1 - x^0), & x^0 \leqslant x < x^1 \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The B-spline $Q(\cdot \mid x^0, ..., x^n)$ is positive on (x^0, x^n) and zero on $(-\infty, x^0) \cup [x^n, \infty)$. On each subinterval (x^i, x^{i+1}) it is an element of π_{n-1} (the space of polynomials of degree $\leqslant n-1$).

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We need some notation for the multivariate case. For any set $A \subset R^n$ let $[A]$ and $\mathrm{vol}_n A$ denote the closed convex hull and the n-dimensional volume of A, respectively. Let $S^n = \{v_0, ..., v_n) \mid \sum_{j=0}^n v_j = 1, \quad v_j \geq 0, j = 0, ..., n\}$ be the standard n-dimensional simplex. Let π_n^s be the space of polynomials of total degree $\leq n$ in s variables.

We can now define the multivariate polynomial B-spline introduced in de Boor [2]. Let $n > s \geq 1$ and let $x^0, ..., x^n$ be any points in R^s such that $\mathrm{vol}_s[x^0, ..., x^n] > 0$. Let $\tilde{x}^0, ..., \tilde{x}^n$ be arbitrary vectors in R^{n-s} for which the denominator in (3) is positive. The multivariate polynomial B-spline, $M(\cdot \mid x^0, ..., x^n)$, corresponding to the knots $x^0, ..., x^n$ is defined by

$$M(x \mid x^0, ..., x^n) = \frac{\mathrm{vol}_{n-s}\{\tilde{x} \in R^{n-s} \mid (x, \tilde{x}) \in [(x^0, \tilde{x}^0), ..., (x^n, \tilde{x}^n)]\}}{\mathrm{vol}_n[(x^0, \tilde{x}^0), ..., (x^n, \tilde{x}^n)]}, \quad (3)$$

where we have chosen the normalization of Micchelli [8] (cf. de Boor [2], Micchelli [7]). This B-spline is a member of π_{n-s}^s in each region bounded by, but not cut by the convex hull of any subsets of s points from $\{x^0, ..., x^n\}$. If every subset of d points of $\{x^0, ..., x^n\}$ forms a convex set of positive volume then $M(\cdot \mid x^0, ..., x^n) \in C^{n-d}(R^s)$. So usually, $M(\cdot \mid x^0, ..., x^n) \in C^{n-s-1}(R^s)$. The support of this function is equal to $[x^0, ..., x^n]$. If we define $\mathrm{vol}_0\, \varnothing = 0$ and $\mathrm{vol}_0 A = 1$ when $A \neq \varnothing$, then we can use (3) to define the multivariate B-spline even in the case $n = s$.

Micchelli [7] discovered a recurrence relation for the evaluation of (3). Suppose that $n > s$ and that $\mathrm{vol}_s[x^0, ..., x^{j-1}, x^{j+1}, ..., x^n] > 0$, $j = 0, ..., n$. Let $x = \sum_{j=0}^n \lambda_j x^j$, where $\lambda_0, ..., \lambda_n$ satisfy $\sum_{j=0}^n \lambda_j = 1$. If $M(\cdot \mid x^0, ..., x^n)$ and $M(\cdot \mid x^0, ..., x^{j-1}, x^{j+1}, ..., x^n)$, $j = 0, ..., n$, are continuous at x, then

$$M(x \mid x^0, ..., x^n) = \frac{n}{n-s} \sum_{j=0}^n \lambda_j M(x \mid x^0, ..., x^{j-1}, x^{j+1}, ..., x^n). \quad (4)$$

An analog recurrence relation is given for the multivariate trigonometric B-splines in Section 3.

To derive (4), Micchelli [7] used the following characterization of the multivariate B-spline.

$$\int_{R^s} M(x \mid x^0, ..., x^n) f(x)\, dx = n! \int_{S^n} f\left(\sum_{j=0}^n v_j x^j\right) dv_1 \cdots dv_n, \qquad \forall f \in C(R^s). \quad (5)$$

In Section 3 a formula resembling this is given for the trigonometric case.

2. A Connection Between Univariate Trigonometric and Polynomial B-Splines

Lyche and Winther [6] defined univariate trigonometric B-splines of arbitrary orders and showed that they possess the recurrence relation

$$T(x \mid x^0, ..., x^n) = \frac{\sin(x - x^0)/2 \; T(x \mid x^0, ..., x^{n-1}) + \sin(x^n - x)/2 \; T(x \mid x^1, ..., x^n)}{\sin(x^n - x^0)/2}. \tag{6}$$

Here $T(\cdot \mid x^0, ..., x^n)$ is the trigonometric B-spline corresponding to the knots $x^0 \leqslant x^1 \leqslant \cdots \leqslant x^n$, where we suppose that $x^n - x^0 < 2\pi$. For $n = 1$,

$$T(x \mid x^0, x^1) = \begin{cases} 1/\sin \dfrac{x^1 - x^0}{2}, & x^0 \leqslant x < x^1 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

$T(\cdot \mid x^0, ..., x^n)$ is positive on (x^0, x^n) and is zero outside of $[x^0, x^n)$. On each subinterval (x^i, x^{i+1}) it is an element of τ_{n-1}, where

$$\tau_r = \begin{cases} \operatorname{span}\{1, \sin x, \cos x, ..., \sin mx, \cos mx\}, & r = 2m \\ \operatorname{span}\{\cos \frac{x}{2}, \sin \frac{x}{2}, ..., \cos(m + \frac{1}{2})\, x, \sin(m + \frac{1}{2})\, x\}, & r = 2m + 1 \end{cases} \tag{8}$$

(cf. Lyche and Winther [6]).

Another basis for τ_r is given by

$$\cos^{r-i}\left(\frac{x}{2}\right) \cdot \sin^i\left(\frac{x}{2}\right), \qquad i = 0, ..., r. \tag{9}$$

The following transformation will be important to us.

$$(\gamma_r f)(x) = \cos^r\left(\frac{x}{2}\right) \cdot f\left(\tan \frac{x}{2}\right). \tag{10}$$

Using $\cos^{r-i}(x/2) \sin^i(x/2) = \cos^r(x/2) \tan^i(x/2)$, $i = 0, ..., r$, we see that γ_r maps π_r onto τ_r. More accurately, if $p \in \pi_r$ on $[\tan(\alpha/2), \tan(\beta/2)$, then $\gamma_r p \in \tau_r$ on $[\alpha, \beta)$, when $-\pi < \alpha < \beta < \pi$. The transformation (10) preserves smoothness, so $\gamma_{n-1} Q(\cdot \mid \tan(x^0/2), ..., \tan(x^n/2))$ should be a trigonometric spline. The following lemma tells us that it is proportional to the trigonometric B-spline $T(\cdot \mid x^0, ..., x^n)$.

LEMMA. *Let $n \in N$ and suppose that $-\pi < x^0 \leqslant x^1 \leqslant \cdots \leqslant x^n < \pi$. Then*

$$T(x \mid x^0, ..., x^n) = \frac{\cos^{n-1}(x/2)}{\prod_{j=0}^{n} \cos(x^j/2)} Q\left(\tan\frac{x}{2} \,\middle|\, \tan\frac{x^0}{2}, ..., \tan\frac{x^n}{2}\right), \qquad x \in (-\pi, \pi). \tag{11}$$

Proof. The proof is by induction on n. The formula

$$\tan A - \tan B = \sin(A-B)/(\cos A \cdot \cos B) \tag{12}$$

will be useful. For $n = 1$, the definitions (2), (7), (10) combined with (12) give (11). Suppose that (11) holds for $n = m$. Let $n = m+1$. If $x^0 = x^n$ then both sides of (11) are identically zero. Suppose $x^0 < x^n$. By (1), (10), (12), and the induction hypothesis, the right side of (11) equals the right side of (6). Hence (11) holds for $n = m+1$. ∎

By the lemma, we may use the more efficient recurrence relation (11) to evaluate $T(x \mid x^0, ..., x^n)$, instead of using (6).

3. MULTIVARIATE TRIGONOMETRIC *B*-SPLINES

In this section we will construct smooth functions of local support which are multivariate trigonometric polynomials in each subdomain. But first we must specify what we mean by a multivariate trigonometric polynomial.

DEFINITION 1. Let $s, n \in N$. Set

$$\left.\begin{aligned} t(x) &= \left(\tan\frac{x_1}{2}, ..., \tan\frac{x_s}{2}\right)^T \\ c(x) &= \prod_{j=1}^{s} \cos\frac{x_j}{2} \end{aligned}\right\}, \qquad x = (x_1, ..., x_s)^T \in (-\pi, \pi)^s. \tag{13}$$

Define

$$\tau_n^s = \{c^n \cdot (p \circ t) \mid p \in \pi_n^s\}, \tag{14}$$

where $p \circ t$ is the composite function of p and t.

By Definition 1,

$$\tau_n^s = \operatorname{span}\left\{\prod_{j=1}^{s} \left(\cos^{n-\alpha_j}\left(\frac{x_j}{2}\right) \cdot \sin^{\alpha_j}\left(\frac{x_j}{2}\right)\right) \,\middle|\, \sum_{j=1}^{s} \alpha_j \leqslant n, \alpha_j \geqslant 0, j = 1, ..., s\right\}. \tag{15}$$

Here we have replaced $x_j^{\alpha_j}$ by the α_jth element in the sequence (9) of basis functions for τ_n. Particularly, $\tau_n^1 = \tau_n$. For $s \geqslant 1$, τ_n^s is contained in the tensor product of s copies of τ_n. By (14) we see that τ_n^s and π_n^s have the same dimension. Thus even if τ_n^s is a space of trigonometric functions, it possesses properties similar to those of π_n^s.

Let us now introduce the multivariate trigonometric B-splines.

DEFINITION 2. Let $n \geqslant s$ and let $x^0, ..., x^n \in (-\pi, \pi)^s$. Suppose that $\operatorname{vol}_s[t(x^0), ..., t(x^n)] > 0$. Define

$$T(x \mid x^0, ..., x^n) = \begin{cases} \dfrac{[c(x)]^{n-s}}{\prod_{j=0}^n c(x^j)} \cdot M(t(x) \mid t(x^0), ..., t(x^n)), & x \in (-\pi, \pi)^s \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where c, t are given by Definition 1 and $M(\cdot \mid t(x^0), ..., t(x^n))$ by (3).

By the lemma, (16) generalizes trigonometric B-splines.

The main result is the following:

THEOREM. *Let $n \geqslant s$ and let $x^0, ..., x^n \in (-\pi, \pi)^s$. Set $y^i = t(x^i)$, $i = 0, ..., n$. Suppose that $\operatorname{vol}_s[y^0, ..., y^n] > 0$. If every subset of d points of $\{y^0, ..., y^n\}$ forms a convex set of positive volume then $T(\cdot \mid x^0, ..., x^n) \in C^{n-d}(R^s)$. For each region D bounded by, but not cut by the convex hull of any subsets of s points from $\{y^0, ..., y^n\}$, $T(\cdot \mid x^0, ..., x^n)_{|t^{-1}(D)} \in \tau_{n-s}^s$. Outside $t^{-1}[y^0, ..., y^n]$, $T(\cdot \mid x^0, ..., x^n)$ is identically zero. Suppose that $n > s+1$ and that $\operatorname{vol}_s[y^0, ..., y^{i-1}, y^{i+1}, ..., y^n] > 0$, $i = 0, ..., n$. Let $\varphi(x) = c(x)\, t(x)$, $x \in (-\pi, \pi)^s$. If the real numbers $\tau_0, ..., \tau_n$ satisfy*

$$\sum_{i=0}^n \tau_i \varphi(x^i) = \varphi(x), \qquad \sum_{i=0}^n \tau_i c(x^i) = c(x) \tag{17}$$

and if $T(\cdot \mid x^0, ..., x^n)$ and $T(\cdot \mid x^0, ..., x^{i-1}, x^{i+1}, ..., x^n)$, $i = 0, ..., n$, are continuous at x, then

$$T(x \mid x^0, ..., x^n) = \frac{n}{n-s} \sum_{i=0}^n \tau_i T(x \mid x^0, ..., x^{i-1}, x^{i+1}, ..., x^n). \tag{18}$$

Proof. The idea behind the proof is to transform properties of $M(\cdot \mid y^0, ..., y^n)$ into properties for $T(\cdot \mid x^0, ..., x^n)$ via Definition 2. The statement about the smoothness of $T(\cdot \mid x^0, ..., x^n)$ follows from what we know about the smoothness of $M(\cdot \mid y^0, ..., y^n)$. Also, $M(\cdot \mid y^0, ..., y^n)_{|D} \in \pi_{n-s}^s$, when D is one of those regions mentioned in the theorem. By Definitions 1 and 2, $T(\cdot|x^0, ..., x^n)|_{t^{-1}(D)} \in \tau^s_{n-s}$. Since $M(\cdot|y^0, ..., y^n)=0$ outside $[y^0, ..., y^n]$, $T(\cdot|x^0, ..., x^n)=0$ outside $t^{-1}[y^0, ..., y^n]$. Let $\tau_0, ..., \tau_n$ satisfy (17). Set $\lambda_i = \tau_i c(x^i)/c(x)$, $i=0, ..., n$. By (17), $\sum_{i=0}^n \lambda_i = 1$ and $\sum_{i=0}^n \lambda_i y^i = t(x)$. Hence (4) is fulfilled when x^i is replaced by y^i and x is replaced by $t(x)$. Multiplying both sides of (4) by $c(x)^{n-s}/\prod_{j=0}^n c(x^j)$ and using Definition 2 we obtain (18). ∎

If the vectors $x^0, ..., x^n$ are in general position, then by the theorem, $d=s+1$, so $T(\cdot|x^0, ..., x^n) \in C^{n-s-1}(R^s)$.

By setting $n-s$ of $\tau_0, ..., \tau_n$ equal to zero we may solve (17) uniquely for the rest of $\tau_0, ..., \tau_n$. As a function of x these τ_j's become elements of τ^s_1.

Using Definition 2 the characterization (5) transforms into

$$\int_{R^s} T(x|x^0, ..., x^n)\, f(x)\, dx = \int_{R^n} \omega(v)\, f(Xv)\, dv_1 \cdots dv_n, \qquad \forall f \in C(R^s), \tag{19}$$

where

$$\omega(v) = \frac{2^s \cdot n!}{\prod_{j=0}^n c(x^j)} \cdot c\left(t^{-1}\left(\sum_{i=0}^n v_i t(x^i)\right)\right)^{n-s+2} \cdot \chi_{S^n}(v),$$
$$Xv = t^{-1}\left(\sum_{i=0}^n v_i t(x^i)\right), \tag{20}$$

and where χ_{S^n} is the characteristic function on S^n. Dahmen and Micchelli [5, Eq. (2.1.4)] consider functions M satisfying (19) for different choices of ω. They always assume that χ is an affine function. This is one of the main differences.

In the same manner other properties of the multivariate polynomial *B*-splines may be transformed into properties for the multivariate trigonometric *B*-splines.

4. Extensions

The conclusions of the theorem hold for more general classes of multivariate *B*-splines. In fact, since we did not use (13) explicitly in the proof of the theorem, we may use any sufficiently smooth mappings t and c in (14) defined on a common domain $\mathscr{D}$ in R^s such that t is injective and c is positive on that domain. We keep (14) as it is and only change $(-\pi, \pi)^s$ into $\mathscr{D}$ in (16). The conclusions of the theorem are still true. As an example, we could use $t(x) = (\tanh x_1, ..., \tanh x_s)^T$, $c(x) = \prod_{j=1}^s \cosh x_j$, $x = (x_1, ..., x_s)^T \in \mathscr{D} = R^s$, and obtain a multivariate hyperbolic *B*-spline (cf. Schumaker [9] for the univariate case). More generally, we could mix cases and use one function class for one variable and a different one for another variable.

Another generalization would be to use, for instance, the box spline in (16) instead of the multivariate polynomial B-spline (see de Boor and DeVore [3], de Boor and Höllig [4]). This would yield a theorem similar to that in Section 3.

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